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The two-component neutrino field in general relativity

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Abstract. The spinor equations for two-component neutrinos in Riemannian space–time are shown to be equivalent to tensor equations analogous to those for an electromagnetic field with a complex current density vector. The structure of the energy–momentum tensor for neutrinos is more complicated than for the Maxwell field, and in constructing exact solutions of the combined neutrino–gravitational equations it has been found necessary to build up the energy–momentum tensor directly from spinor quantities. Exact solutions for pure radiation fields are obtained in the case of conformally flat and cylindrically symmetric space–times, and also for a non-diagonal radiation metric. It is shown that there are no spherically symmetric radiation solutions for the type of neutrino field considered.

1. Introduction

In recent years much work has been done on neutrino fields, a particular interest in two-component theory having been stimulated by Lee and Yang (1956). Most of this work has been concerned with attempts to obtain a geometrical theory for a neutrino field, or to consider and compare different methods of adapting spinor equations to general relativity. However, very few exact solutions have been obtained, and those that have been published are usually comparatively trivial. The aim of this paper is to examine the tensor representation of the neutrino equations and to develop a method of obtaining solutions, which can be interpreted as two-component neutrino fields, by solving directly the field equations for the combined neutrino-gravitational field subject to the neutrino condition.

In order to build up a mathematical description of a neutrino field, we start with Dirac's equation for a relativistic particle in Minkowski space–time

$$\gamma^{\mu}\left(-\mathrm{i}\hbar\partial_{\mu}+\frac{e}{c}A_{\mu}\right)\phi+mc\phi = 0$$

where the γ^{μ} are a set of 4×4 matrices which obey

$$\gamma^{\mu}\gamma^{\nu}+\gamma^{\nu}\gamma^{\mu} = 2\eta^{\mu\nu}I \tag{1.1}$$

where

$$\eta^{\mu\nu} = \begin{pmatrix}1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1\end{pmatrix}$$

and ϕ is a four-component spinor. For our purposes we can define a neutrino to be that particle which is described by Dirac's equation and has zero mass and zero charge. Thus it obeys

$$\gamma^{\mu}\phi_{,\mu} = 0 \tag{1.2}$$

in flat space–time. We use a comma to denote partial differentiation and a semi-colon to denote covariant differentiation.

In § 2 we indicate the method of generalizing Dirac's equation by using four matrices γ^{μ} which obey $\gamma^{\mu}\gamma^{\nu}+\gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}I$ at every point. The covariant derivative of spinors in Riemannian space–time is then considered in § 3.

Then in § 4 we develop the tensor method of incorporating Dirac's equation into general relativity. Whittaker (1937) and Penney (1965) have obtained a connection between a two-component spinor and a self-dual antisymmetric tensor $S_{\mu\nu}$ of zero norm. They have obtained the identities

$$S^{\mu\alpha}S_{\alpha\nu} = 0 \tag{1.3}$$

$$S^{\mu\alpha}S^{*}_{\alpha\nu} = D^{\mu}D_{\nu} \tag{1.4}$$

$$S^{\mu\alpha}D_{\alpha;\nu} = D^{\mu}H_{\nu} \tag{1.5}$$

where the asterisk denotes the conjugate complex, D_{μ} is a real null vector and H_{μ} is in general complex. In this paper we use the complex dual defined by

$$\hat{S}_{\mu\nu} = E_{\mu\nu\alpha\beta}S^{\alpha\beta}$$

where

$$E_{\mu\nu\alpha\beta} = \tfrac{1}{2}\sqrt{g}[\mu\nu\alpha\beta]$$

$[\mu\nu\alpha\beta]$ being the permutation symbol and g the determinant of the metric tensor. We then show that the neutrino equation in Riemannian space–time is equivalent to the tensor equation

$$S^{\mu\nu}{}_{;\nu} = H^{\mu}. \tag{1.6}$$

The neutrino equations (1.3), (1.4) and (1.6) are remarkably similar to corresponding equations for the null electromagnetic field. With suitable units the Maxwell equations may be expressed in the form (Goodinson and Newing 1968)

$$\omega^{\mu\nu}{}_{;\nu} = j^{\mu}$$

where j^{μ} is the current density vector and the complex self-dual tensor $\omega^{\mu\nu}$ is defined in terms of the electromagnetic field tensor $F^{\mu\nu}$ by

$$\omega^{\mu\nu} = F^{\mu\nu} + \hat{F}^{\mu\nu}.$$

For a null field, i.e. a field for which the two invariants $F_{\alpha\beta}F^{\beta\alpha}$ and $F_{\alpha\beta}\hat{F}^{\beta\alpha}$ are zero, $\omega_{\mu\nu}$ is such that

$$\omega^{\mu\alpha}\omega_{\alpha\nu} = 0$$

and

$$\omega^{\mu\alpha}\omega^{*}_{\alpha\nu} = L^{\mu}L_{\nu}$$

where L_{μ} is the propagation vector of the electromagnetic field. The neutrino and Maxwell fields differ in that the energy–momentum tensor $E_{\mu}{}^{\nu}$ for the Maxwell field is given by $\omega_{\mu\alpha}\omega^{*\alpha\nu}$, whereas it has not been possible to find a simple expression for $E_{\mu}{}^{\nu}$ in terms of the tensor $S_{\mu\nu}$ in the case of the neutrino field. For this reason it has been found necessary to build up $E_{\mu\nu}$ from spinor quantities when constructing exact solutions of the combined neutrino–gravitational field equations.

We then consider a purely spinor approach in the last section and obtain a number of exact solutions of the neutrino–gravitational field equations corresponding to the particular case of pure radiation fields. Solutions are obtained which correspond to a non-diagonal radiation metric, to conformally flat space–time and to a cylindrically symmetric space–time. Finally, it is shown that a pure radiation field which is spherically symmetric cannot be interpreted as two-component neutrino radiation.

2. Dirac matrices and bilinear covariants

An obvious way of generalizing the commutation relations (1.1) to Riemannian space–time is to introduce new matrices γ^{μ} such that

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}I. \tag{2.1}$$

These new matrices can be represented by a linear combination of the matrices of special relativity. In this paper we consider only those γ matrices which are linear combinations of the four constant matrices β_μ of Dirac's standard representation in Minkowski space:

$$\beta_0 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \ \beta_1 = \begin{pmatrix} & & & -1 \\ & & -1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}, \ \beta_2 = \begin{pmatrix} & & & \mathrm{i} \\ & & -\mathrm{i} & \\ & -\mathrm{i} & & \\ \mathrm{i} & & & \end{pmatrix}, \ \beta_3 = \begin{pmatrix} & & -1 & \\ & & & 1 \\ 1 & & & \\ & -1 & & \end{pmatrix}.$$

Then

$$\beta^5 = \mathrm{i}\beta^0\beta^1\beta^2\beta^3 = \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix}, \quad C = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ -1 & & & \end{pmatrix} \tag{2.2}$$

where C is Schwinger's C matrix (defined by $\tilde{\gamma}^\mu = -C\gamma^\mu C^{-1}$).

The neutrino equation (1.2) can thus be carried over into general relativity and becomes

$$\gamma^\alpha \phi_{|\alpha} = 0 \tag{2.3}$$

where $\phi_{|\alpha}$ is the covariant derivative of ϕ in spinor space. Since the γ^μ are linear combinations of the β^μ, the Hermitian conjugate of the γ^μ can be obtained by

$$\gamma_\mu{}^\dagger = \beta_0 \gamma_\mu \beta_0.$$

If $\tilde{\phi}$ denotes the transpose of a spinor and $\phi^\dagger$ its Hermitian conjugate, we define the adjoint and the charge conjugate spinors by

$$\bar{\phi} = \phi^\dagger \beta_0$$
$$\phi^c = C\tilde{\bar{\phi}}.$$

With a suitable definition of spinor differentiation, (2.3) implies that

$$\bar{\phi}_{|\alpha}\gamma^\alpha = 0. \tag{2.4}$$

We also need to define the matrix

$$\gamma^5 = \tfrac{1}{12} E_{\kappa\lambda\mu\nu}\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu$$

which anti-commutes with all the matrices γ^μ and has the property

$$(\gamma^5)^2 = 1.$$

Also, since γ^μ are linear combinations of β^μ, $\gamma^5 = \beta^5$; the proof of this is given in appendix 1.

From these quantities it is possible to construct the bilinear covariants $\bar{\psi}\gamma_\mu\phi$, $\bar{\psi}\gamma_\mu\gamma_\nu\phi$, $\bar{\psi}\gamma_\mu\gamma^5\phi$ etc. Pauli (1935, 1936) and Kofink (1937, 1940) have obtained a number of identities involving products of these covariants. In this paper we make use only of the identity

$$\begin{aligned}(\bar{\theta}P\gamma^\mu\phi)(\bar{\psi}Q\gamma_\mu\chi) = {} & (\bar{\theta}P\chi)(\bar{\psi}Q\phi) - (\bar{\theta}P\beta^5\chi)(\bar{\psi}Q\beta^5\phi) \\ & - (\bar{\theta}P\beta^0\chi)(\bar{\psi}Q\beta_0\phi) + (\bar{\theta}P\beta^0\beta^5\chi)(\bar{\psi}Q\beta^5\beta_0\phi) \\ & + (\bar{\theta}P\beta^0\phi)(\bar{\psi}Q\beta_0\chi) - (\bar{\theta}P\beta^0\beta^5\phi)(\bar{\psi}Q\beta^5\beta_0\chi)\end{aligned} \tag{2.5}$$

where θ, ϕ, ψ, χ are any spinors and P and Q are arbitrary matrices. This identity is simply a restatement of Kofink's identity (1940, p. 438).

3. Covariant differentiation of spinors

Following a common notation (see, for example, Brill and Wheeler 1957) we shall define the covariant derivative of a spinor by

$$\phi_{|\nu} = \phi_{,\nu} - \Gamma_\nu \phi \tag{3.1}$$

$$\bar{\phi}_{|\nu} = \bar{\phi}_{,\nu} + \bar{\phi}\Gamma_\nu \tag{3.2}$$

where the spinor connection Γ_ν will be expected to depend on the metric tensor as well as on the β matrices. We also require that the γ matrices be constant with respect to spinor differentiation and that the spinor derivations of bilinear covariants are the same as the corresponding covariant tensor derivatives. The requirement that $(\bar{\phi}^c \gamma_\lambda \gamma_\mu \phi)_{;\nu} = (\bar{\phi}^c \gamma_\lambda \gamma_\mu \phi)_{|\nu}$ implies that

$$\tilde{\Gamma}_\nu = -C\Gamma_\nu C^{-1}$$

and this condition eliminates the arbitrary vector field which is sometimes associated with the spinor connection. The connection Γ_ν is then uniquely determined and, with the two-component restriction $\phi = -\gamma^5\phi$ referred to later, Γ_ν may be shown to be

$$\Gamma_\nu = \tfrac{1}{4}\gamma_{\alpha;\nu}\gamma^\alpha.$$

4. Tensor equation for the two-component neutrino field†

Whittaker (1937) and Penney (1965) have shown that any two-component spinor field has an associated set of tensors D_μ, $S_{\mu\nu}$, H_μ, where D_μ is a real vector, $S_{\mu\nu}$ is a self-dual antisymmetric tensor and H_μ is complex, which satisfy

$$S^{\mu\alpha} S^*{}_{\alpha\nu} = D^\mu D_\nu \tag{4.1}$$

$$S^{\mu\alpha} D_{\alpha;\nu} = D^\mu H_\nu \tag{4.2}$$

$$D_\alpha D^\alpha = 0, \qquad S_{\mu\alpha} S^{\alpha\nu} = 0. \tag{4.3}$$

These tensors can be related to non-zero bilinear covariants of the spinor field. In fact, if we make the two-component restriction

$$\phi = -\gamma^5\phi \tag{4.4}$$

then the tensors can be identified by

$$D_\mu = \bar{\phi}\gamma_\mu\phi$$

$$S_{\mu\nu} = \frac{1}{\sqrt{2}}\bar{\phi}^c\gamma_\mu\gamma_\nu\phi$$

$$H_\mu = \frac{1}{\sqrt{2}}(\bar{\phi}^c_{|\mu}\phi - \bar{\phi}^c\phi_{|\mu}).$$

$S_{\mu\nu}$ is self-dual on account of (4.4) and (A1), and the identities (4.1), (4.2) and (4.3) can be shown to be satisfied on account of the Pauli–Kofink identity (2.5). The vector D_μ is interpreted quantum-mechanically as a probability current, but will be interpreted here classically as a neutrino flux. It follows from the neutrino equation (2.3) that

$$D^\alpha{}_{|\alpha} = 0. \tag{4.5}$$

Taking the divergence of $S^{\mu\nu}$, we obtain

$$S^{\mu\nu}{}_{|\nu} = H^\mu + \sqrt{2}\bar{\phi}^c\gamma^\mu\gamma^\nu\phi_{|\nu}.$$

† Tensor equivalents of the four-component Dirac field with non-zero rest mass have been extensively discussed by Yamamoto (1936).

The neutrino equation (2.3) implies that

$$S^{\mu\nu}{}_{|\nu} = H^{\mu}. \tag{4.6}$$

This can also be shown to be a sufficient condition for the neutrino field by considering the product $(\bar{\psi}\gamma_\alpha\chi)(\bar{\phi}^c\gamma^\alpha\gamma^\nu\phi_{|\nu})$, where ψ and χ are arbitrary spinors with the only restriction that

$$\chi = -\gamma^5\chi.$$

Now, using the Pauli–Kofink identity (2.5), we obtain

$$(\bar{\psi}\gamma_\alpha\chi)(\bar{\phi}^c\gamma^\alpha\gamma^\nu\phi_{|\nu}) = 2(\bar{\psi}\gamma^\nu\phi_{|\nu})(\bar{\phi}^c\chi).$$

Thus, since ψ and χ are arbitrary, (4.6) implies the vanishing of $\gamma^\nu\phi_{|\nu}$, and is therefore a necessary and sufficient condition for a neutrino field (Goodinson 1969).

Equation (4.6) is analogous to the Maxwell equations $\omega^{\mu\nu}{}_{;\nu} = j^\mu$ for a null electromagnetic field, H_μ playing the role of a current vector. This analogy was noticed by Penney (1965), but he confined his attention to the case $H_\mu = 0$. For the general neutrino field H_μ will be a complex non-zero vector.

The field equations for a combined neutrino–gravitational field are (4.6) or (2.3), together with the Einstein gravitational equations

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = -E_{\mu\nu}$$

where the energy momentum tensor for the neutrino field may be taken to be (Pauli 1933, Brill and Wheeler 1957)

$$E_{\mu\nu} = \tfrac{1}{4}\mathrm{i}c\hbar\{\bar{\phi}_{|\mu}\gamma_\nu\phi - \bar{\phi}\gamma_\nu\phi_{|\mu} + \bar{\phi}_{|\nu}\gamma_\mu\phi - \bar{\phi}\gamma_\mu\phi_{|\nu}\}. \tag{4.7}$$

It follows from (2.3) that $E_\alpha{}^\alpha = 0$, just as for the electromagnetic field. The curvature scalar R is therefore zero, and in suitable units the gravitational equations reduce to

$$R_{\mu\nu} = -E_{\mu\nu}. \tag{4.8}$$

For pure radiation fields it is reasonable to suppose that

$$E^{\mu\nu} = \sigma^2 D^\mu D^\nu \tag{4.9}$$

where σ is some real scalar. In this case the divergence condition $R^{\mu\nu}{}_{;\nu} = 0$ implies that

$$2\sigma_{,\nu}D^\nu D^\mu + \sigma D^\mu{}_{;\nu}D^\nu = 0 \tag{4.10}$$

on account of (4.5). This equation could be satisfied by taking σ constant and $D_{\mu;\nu} = 0$. This case leads us to the condition on the Ricci tensor $R_{\lambda\mu;\nu} = 0$ which is obtained by Penney (1965) and by Inomata and McKinley (1965). Neutrino fields satisfying this condition are the 'restricted class' defined by the latter authors.

In conclusion, it may be noticed that on account of (4.9) and (4.10) less restrictive necessary conditions for the existence of a pure neutrino radiation field are provided by the Rainich-type conditions

$$R_\alpha{}^\alpha = 0, \qquad R_{\mu\alpha}R^{\alpha\nu} = 0$$

$$R_{\mu\nu;\alpha}R^\alpha{}_\rho = R_{\rho\nu;\alpha}R^\alpha{}_\mu.$$

If σ is a constant the latter condition may be replaced by

$$R_{\mu\nu;\alpha}R^{\alpha\rho} = 0.$$

5. Some exact solutions of the neutrino–gravitational equations

The tensor technique developed in § 4 is not complete in that it is not at all obvious that the expression $E_{\mu\nu} = \sigma^2 D_\mu D_\nu$ assumed for the energy–momentum tensor of the neutrino radiation field will be consistent with the definition (4.7) of $E_{\mu\nu}$ in terms of the spinor field. It is therefore necessary, in constructing exact solutions of the neutrino–gravitational field, to take the spinors as the fundamental quantities. The basic equations of the combined field are the neutrino spinor equation (2.3), the definition of $E_{\mu\nu}$ in terms of spinors (4.7) and the standard relativistic field equations (4.8). The two-component restriction $\phi = -\gamma^5\phi$ implies that ϕ is of the form $\phi = \begin{bmatrix} u \\ -u \end{bmatrix}$, where $u = \begin{bmatrix} p \\ q \end{bmatrix}$.

5.1. *Solution* (1)

We shall first consider neutrino fields admitted by space–times with metrics of the form

$$\mathrm{d}s^2 = h_0{}^2(\mathrm{d}x^0)^2 + 2\mathrm{d}x^0\,\mathrm{d}x^1 - h_2{}^2\{(\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2\}. \tag{5.1}$$

For this metric the γ matrices may be taken to be

$$\gamma_0 = h_0\beta_0, \qquad \gamma_1 = \frac{1}{h_0}(\beta_0 + \beta_1), \qquad \gamma_2 = h_2\beta_2, \qquad \gamma_3 = h_2\beta_3. \tag{5.2}$$

For this metric we are taking x^1 to be a radiation coordinate. Thus the vector D_μ is given by $D_\mu = D_0\delta^0_\mu$ and this (see appendix 2) implies that

$$p = -q. \tag{5.3}$$

Let us consider the neutrino field defined by (5.1), (5.2) and (5.3) with the restriction of axial symmetry, i.e. h_0, h_2 and p are independent of x^3. Expressions for the Ricci tensor, the energy–momentum tensor, the neutrino condition and the vector H_μ are given in appendix 2.

For axial symmetry $R_{03} = 0$, and so, for non-zero p, h_2 must be independent of x^2. Then, since $R_{11} = 0$ we could take

$$h_2 = x^1 f(x^0)$$

where $f(x^0)$ is arbitrary.

The neutrino condition implies that p is of the form

$$p = \frac{A(x^0)}{x^1\sqrt{h_0}}$$

where $A(x^0)$ is an arbitrary complex function. This implies that E_{01} and E_{02} are both zero, and these in turn give

$$(h_0{}^2)_{,1} = 2\frac{f_{,0}}{f} + \frac{2m(x^0)}{(x^1)^2}$$

where $m(x^0)$ is arbitrary. Then $R_{22} = 0$ gives the final condition that

$$h_0{}^2 = 2x^1\frac{f_{,0}}{f} - \frac{2m}{x^1}.$$

The Ricci tensor for the metric

$$\mathrm{d}s^2 = \left(\frac{2x^1 f_{,0}}{f} - \frac{2m}{x^1}\right)(\mathrm{d}x^0)^2 + 2\,\mathrm{d}x^0\,\mathrm{d}x^1 - (x^1)^2 f^2\{(\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2\}$$

is

$$R_{\mu\nu} = \frac{2m}{(x^1)^2}\left(\frac{3f_{,0}}{f} + \frac{m_{,0}}{m}\right)\delta_\mu{}^0\delta_\nu{}^0$$

and the neutrino–gravitational equations are satisfied if the neutrino flux vector is taken to be $D_\mu = 4h_0 pp^*\delta_\mu{}^0$, where

$$p^2 = \frac{-m}{2c\hbar\xi_{,0}(x^1)^2 h_0}\left(\frac{3f_{,0}}{f} + \frac{m_{,0}}{m}\right)e^{2i\xi}$$

ξ being an arbitrary function of x^0, the Ricci tensor then being $R_{\mu\nu} = -\sigma^2 D_\mu D_\nu$, where

$$\sigma = c\hbar x^1\xi_{,0}\left(-6m\frac{f_{,0}}{f} - 2m_{,0}\right)^{-1/2}.$$

We are indebted to a referee for the observation that it is possible to make a coordinate transformation such that f can be taken to be unity.

5.2. *Solution* (2)

Now consider the similar metric given by

$$ds^2 = B(x^0, x^2, x^3)(dx^0)^2 + 2dx^0\,dx^1 - (dx^2)^2 - (dx^3)^2.$$

Expressions for the Ricci tensor, the energy momentum tensor and the neutrino condition can again be obtained from appendix 2. The neutrino condition gives

$$pB^{1/4} = A(x^0)\exp\{L(x^0)M(x^2 - ix^3)\}$$

where A, L and M are arbitrary, L and M being complex. But $E_{02} = 0$ and $E_{03} = 0$ imply that M = const., and so we can put

$$p = B^{-1/4}P(x^0)\exp\{i\xi(x^0)\}.$$

Then

$$B_{,22} + B_{,33} = 4c\hbar\sqrt{B}i(pp^*_{,0} - p^*p_{,0})$$

$$= 8c\hbar\xi_{,0}P^2$$

which is independent of x^2 and x^3, and so we can write B in the form

$$B = a(x^0)(x^2)^2 + b(x^0)x^2x^3 + c(x^0)(x^3)^2 + d(x^0)x^2 + e(x^0)x^3 + f(x^0)$$

where

$$a(x^0) + c(x^0) = 4c\hbar\xi_{,0}P^2.$$

5.3. *Solution* (3)

Collinson (1968) has shown that the neutrino fields considered by Penney (1965) are necessarily conformally flat. So we consider the metric

$$ds^2 = e^{2\theta}\{(dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2\}. \tag{5.4}$$

The Ricci tensor for this metric is

$$R_{\mu\nu} = 2(\theta_{,\mu\nu} - \theta_{,\mu}\theta_{,\nu}) + g_{\mu\nu}g^{\alpha\beta}(\theta_{,\alpha\beta} + 2\theta_{,\alpha}\theta_{,\beta})$$

and the energy–momentum tensor reduces to

$$E_{\mu\nu} = ic\hbar e^\theta\{(\delta^0_\mu - \delta^1_\mu)(pp^*_{,\nu} - p^*p_{,\nu}) + (pp^*_{,\mu} - p^*p_{,\mu})(\delta^0_\nu - \delta^1_\nu)\}.$$

The neutrino condition becomes

$$\left(\frac{\partial}{\partial x^0}+\frac{\partial}{\partial x^1}\right)(\ln p+\tfrac{3}{2}\theta) = 0$$

$$\left(\mathrm{i}\frac{\partial}{\partial x^2}+\frac{\partial}{\partial x^3}\right)(\ln p+\tfrac{3}{2}\theta) = 0.$$

Thus

$$p = \exp(-\tfrac{3}{2}\theta)\exp\{L(x^0-x^1)M(x^2-\mathrm{i}x^3)\}.$$

Now, to solve the field equations, $R_{\mu\nu} = -E_{\mu\nu}$.

$E_{23} = 0$, $E_{22} = 0$, $E_{33} = 0$ imply that

$$\mathrm{e}^{-\theta} = f(x^0, x^1)+A(x^0, x^1)x^2+B(x^0, x^1)x^3+C(x^0, x^1)\{(x^2)^2+(x^3)^2\}$$

and

$$\mathrm{e}^{\theta}g^{\alpha\beta}(\theta_{,\alpha\beta}+2\theta_{,\alpha}\theta_{,\beta})+4C(x^0, x^1) = 0.$$

The field equations in E_{02}, E_{12}, E_{03} and E_{13} give A, B and C as functions of x^0-x^1. Those involving E_{00}, E_{01}, E_{11} give

$$2f_{,01}+f_{,00}+f_{,11} = 0$$

$$f_{,11}-f_{,00} = 4C$$

$$f_{,00}+A''x^2+B''x^3+C''\{(x^2)^2+(x^3)^2\}+2C = c\hbar\mathrm{i}(pp^*_{,0}-p^*p_{,0}).$$

A general solution of these is given by

$$A = \text{const.},\qquad B = \text{const.},\qquad C = 0$$

$$f = f(x^0-x^1)+Dx^1+(A^2+B^2+D^2)^{1/2}x^0,\qquad D = \text{const}.$$

Thus for a conformally flat space of type (5.4) we have the exact solution

$$\mathrm{e}^{-\theta} = f(x^0-x^1)+a_\mu x^\mu$$

where a_μ are constants satisfying $a_\mu a^\mu = 0$

$$p = \exp(-\tfrac{3}{2}\theta)\exp\{l(x^0-x^1)+\mathrm{i}\lambda(x^0-x^1)\}$$

where $l(x^0-x^1)$ and $\lambda(x^0-x^1)$ are arbitrary real functions which satisfy

$$\lambda'\mathrm{e}^{2l} = \frac{1}{2c\hbar}\mathrm{e}^{3\theta}(\mathrm{e}^{-\theta})_{,00}.$$

This is a radiation solution of type $R_{\mu\nu} = -\sigma^2 D_\mu D_\nu$ where

$$\sigma^2 = \tfrac{1}{4}c\hbar\,\mathrm{e}^{2\theta}\,\mathrm{e}^{-2l}\lambda'$$

and the vector H_μ is given by

$$H_\mu = 2\sqrt{2}p^2\,\mathrm{e}^{\theta}\{(\mathrm{i}a_2+a_3)(\delta^0_\mu-\delta^1_\mu)+(a_0+a_1)(\mathrm{i}\delta^2_\mu+\delta^3_\mu)\}.$$

5.4. *Solution* (4)

Let us consider now a cylindrically symmetric space–time with the metric

$$\mathrm{d}s^2 = \mathrm{e}^{2\theta}(\mathrm{d}t^2-\mathrm{d}r^2)-\mathrm{e}^{2\psi}\{(\mathrm{d}x^2)^2+(\mathrm{d}x^3)^2\} \tag{5.5}$$

where θ and ψ are functions of r and t only. The non-zero components of the Ricci tensor are

$$\begin{aligned}
R_{00} &= \theta_{,00}+2\psi_{,00}-2\psi_{,0}\theta_{,0}+2\psi_{,0}^2-\theta_{,11}-2\psi_{,1}\theta_{,1}\\
R_{01} &= 2\psi_{,01}-2\psi_{,0}\theta_{,1}-2\psi_{,1}\theta_{,0}-2\psi_{,0}\psi_{,1}\\
R_{11} &= \theta_{,11}+2\psi_{,11}-2\psi_{,1}\theta_{,1}+2\psi_{,1}^2-\theta_{,00}-2\psi_{,0}\theta_{,0}\\
R_{22} &= \exp\{2(\psi-\theta)\}(-\psi_{,00}-2\psi_{,0}^2+\psi_{,11}+2\psi_{,1}^2)\\
R_{33} &= R_{22}.
\end{aligned}$$

The energy–momentum tensor for this neutrino field is

$$E_{\mu\nu} = \mathrm{i}c\hbar \mathrm{e}^{\theta}\{(\delta_\mu^0-\delta_\mu^1)(pp^*_{,\nu}-p^*p_{,\nu})+(pp^*_{,\mu}-p^*p_{,\mu})(\delta_\nu^0-\delta_\nu^1)\}.$$

The neutrino condition becomes

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial r}\right)(\ln p^2+\theta+2\psi) = 0$$

$$\left(\mathrm{i}\frac{\partial}{\partial x^2}+\frac{\partial}{\partial x^3}\right)(\ln p^2+2\theta+\psi) = 0.$$

Thus

$$\ln p^2+\theta+2\psi = 2L(r-t)M(x^2-\mathrm{i}x^3)$$

where L and M are arbitrary complex functions. But $E_{02} = 0$ and $E_{03} = 0$ imply that M is constant. Therefore

$$p^2 = \mathrm{e}^{-\theta}\,\mathrm{e}^{-2\psi}\exp\{2L(r-t)\}$$

and therefore $E_{00} = E_{11} = -E_{01}$. These equations provide relations between θ, ψ and their derivatives. To obtain a particular solution of these we take the case when ψ is independent of t and obtain the solution

$$\begin{aligned}
\mathrm{e}^{2\psi} &= r\\
\mathrm{e}^{2\theta} &= \frac{1}{\sqrt{r}}\exp\{2f(r-t)\}
\end{aligned}\tag{5.6}$$

where $f(r-t)$ is arbitrary. We put

$$L(r-t) = l(r-t)+\mathrm{i}\lambda(r-t)$$

where $l+\lambda$ are real functions. Then

$$p = r^{-3/8}\exp\{-\tfrac{1}{2}f(r-t)\}\exp(l+\mathrm{i}\lambda)$$

together with the condition

$$\lambda'\,\mathrm{e}^{2l} = -\frac{1}{4c\hbar}f'.$$

This is also a radiation solution of type $R_{\mu\nu} = -\sigma^2 D_\mu D_\nu$, where

$$\sigma^2 = -\tfrac{1}{4}c\hbar r\,\mathrm{e}^{-2l}\lambda'$$

and the vector H_μ is given by

$$H_\mu = -\sqrt{2}p^2\,\mathrm{e}^{-2f}(\mathrm{i}\delta_\mu^2+\delta_\mu^3).$$

The metric given by (5.5) and (5.6) is a particular case of Rao's (1964) solution (ii).

The associated space–time has also been interpreted as admitting an electromagnetic field (Goodinson 1969).

5.5. *Solution* (5)

The purpose of this paragraph is to demonstrate that it is impossible to obtain a spherically symmetric pure radiation field of the type $R_{\mu\nu} = -\sigma^2 D_\mu D_\nu$ for the two-component neutrino fields considered here. This result is analogous to the extension of Birkhoff's theorem to Einstein–Maxwell fields (Hoffmann 1932). Let us consider the most general spherically symmetric metric

$$\mathrm{d}s^2 = A\,\mathrm{d}t^2 - B\,\mathrm{d}r^2 - C(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)$$

where A, B and C are functions of r and t. The Ricci tensor for this metric has components

$$\begin{aligned} &R_{00}, R_{01}, R_{11}, R_{22} \quad \text{non-zero}\\ &R_{33} = \sin^2\theta R_{22}\\ &R_{02}, R_{03}, R_{12}, R_{13}, R_{23} \quad \text{zero.}\end{aligned}$$

For a spherically symmetric solution it is necessary that $D_2 = 0 = D_3$. So, necessarily $p = \pm q$, in which case the energy–momentum tensor for the neutrino field is

$$\begin{aligned} E_{\mu\nu} = c\hbar\{&(\sqrt{A}\delta^0_\mu \pm \sqrt{B}\delta^1_\mu)(\mathrm{i}p\overset{*}{p}_{,\nu} - \mathrm{i}p^*p_{,\nu} + \cos\theta pp^*\delta^3_\nu)\\ &+(\mathrm{i}p\overset{*}{p}_{,\mu} - \mathrm{i}p^*p_{,\mu} + \cos\theta pp^*\delta^3_\mu)(\sqrt{A}\delta^0_\nu \pm \sqrt{B}\delta^1_\nu)\}.\end{aligned}$$

Now, we put $p = P\mathrm{e}^{\mathrm{i}\xi}$, where P and ξ are real functions of all four coordinates, and consider the field equations $R_{02} = 0$ and $R_{03} = 0$. These give

$$\mathrm{i}c\hbar\sqrt{A}(p\overset{*}{p}_{,2} - p^*p_{,2}) = 0$$

i.e.

$$P^2 2\frac{\partial\xi}{\partial\theta} = 0$$

and

$$\mathrm{i}c\hbar\sqrt{A}(p\overset{*}{p}_{,3} - p^*p_{,3}) + c\hbar\sqrt{A}\cos\theta pp^* = 0.$$

i.e.

$$P^2\left(2\frac{\partial\xi}{\partial\phi} + \cos\theta\right) = 0.$$

The only possible solution of these is that $P = 0$. Thus we have shown that a spherically symmetric pure radiation field cannot be interpreted as neutrino radiation.

The physical significance of the exact solutions obtained here and the question of neutrino fields which are not purely radiational will be considered in a later paper.

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Appendix 1

If we define the complex alternating tensor by

$$E_{\kappa\lambda\mu\nu} = \tfrac{1}{2}\sqrt{g}[\kappa\lambda\mu\nu]$$

and put

$$\gamma^{\alpha\beta} = \tfrac{1}{2}(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha)$$

and

$$\gamma^5 = \tfrac{1}{12}E_{\kappa\lambda\mu\nu}\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu$$

then we have the following relationships between the γ matrices

$$\begin{aligned} \gamma_\rho\gamma^5 &= \tfrac{1}{3}E_{\rho\lambda\mu\nu}\gamma^\lambda\gamma^\mu\gamma^\nu \\ \gamma^{\kappa\lambda} &= -E^{\kappa\lambda\mu\nu}\gamma_{\mu\nu}\gamma^5 \\ \gamma_\kappa\gamma_\lambda\gamma_\mu &= g_{\kappa\lambda}\gamma_\mu - g_{\kappa\mu}\gamma_\lambda + g_{\lambda\mu}\gamma_\kappa - 2E_{\kappa\lambda\mu\nu}\gamma^\nu\gamma^5. \end{aligned} \tag{A1}$$

If the γ^μ are linear combinations of β^μ, the matrices of Dirac's standard representation, then we can put $\gamma^\alpha = h^{\alpha a}\beta_a$. Then

$$\begin{aligned} 2g^{\alpha\beta} &= \gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = h^{\alpha a}h^{\beta b}(\beta_a\beta_b + \beta_b\beta_a) \\ &= 2h^{\alpha a}h^{\beta b}\eta_{ab}. \end{aligned}$$

Thus

$$\frac{1}{g} = h^2\eta.$$

Now

$$\begin{aligned} \gamma^5 &= \frac{1}{24}\sqrt{g}[\alpha\beta\gamma\delta]\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta \\ &= \frac{1}{24}\sqrt{g}[\alpha\beta\gamma\delta]h^{\alpha a}h^{\beta b}h^{\gamma c}h^{\delta d}\beta_a\beta_b\beta_c\beta_d \\ &= \frac{1}{24}\sqrt{g}h[abcd]\beta_a\beta_b\beta_c\beta_d \\ &= \frac{1}{24}\frac{1}{\sqrt{\eta}}[abcd]\beta_a\beta_b\beta_c\beta_d \\ &= \mathrm{i}\beta^0\beta^1\beta^2\beta^3 \end{aligned}$$

i.e.

$$\gamma^5 = \beta^5.$$

Appendix 2

For the metric

$$\mathrm{d}s^2 = h_0{}^2(\mathrm{d}x^0)^2 + 2\mathrm{d}x^0\mathrm{d}x^1 - h_2{}^2\{(\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2\}$$

the components of the Ricci tensor are

$$\begin{aligned} R_{00} = {} & \frac{2h_{2,00}}{h_2} + 2\frac{h_0}{h_2}(h_{0,1}h_{2,0} - h_{0,0}h_{2,1}) - h_0{}^4\left\{\frac{h_{0,11}}{h_0} + \left(\frac{h_{0,1}}{h_0}\right)^2 + 2\frac{h_{0,1}}{h_0}\frac{h_{2,1}}{h_2}\right\} \\ & - \left(\frac{h_0}{h_2}\right)^2\left\{\frac{h_{0,22}}{h_0} + \left(\frac{h_{0,2}}{h_0}\right)^2 + \frac{h_{0,33}}{h_0} + \left(\frac{h_{0,3}}{h_0}\right)^2\right\} \end{aligned}$$

$$R_{01} = \frac{2h_{2,01}}{h_2} - h_0{}^2\left\{\frac{h_{0,11}}{h_0} + \left(\frac{h_{0,1}}{h_0}\right)^2 + 2\frac{h_{0,1}}{h_0}\frac{h_{2,1}}{h_2}\right\}$$

$$R_{02} = \frac{h_{2,02}}{h_2} - \frac{h_{2,0}}{h_2}\frac{h_{2,2}}{h_2} - h_0{}^2\left\{\frac{h_{0,12}}{h_0} + \frac{h_{0,1}}{h_0}\frac{h_{0,2}}{h_0}\right\}$$

$$R_{11} = \frac{2h_{2,11}}{h_2}, \qquad R_{03} = \frac{h_{2,03}}{h_2} - \frac{h_{2,0}}{h_2}\frac{h_{2,3}}{h_2} - h_0^2\left\{\frac{h_{0,13}}{h_0} + \frac{h_{0,1}}{h_0}\frac{h_{0,3}}{h_0}\right\}$$

$$R_{12} = \frac{h_{2,12}}{h_2} - \frac{h_{2,1}}{h_2}\frac{h_{2,2}}{h_2}$$

$$R_{13} = \frac{h_{2,13}}{h_2} - \frac{h_{2,1}}{h_2}\frac{h_{2,3}}{h_2}$$

$$R_{22} = \frac{h_{2,22}}{h_2} - \left(\frac{h_{2,2}}{h_2}\right)^2 + \frac{h_{2,33}}{h_2} - \left(\frac{h_{2,3}}{h_2}\right)^2 - 2h_2^2\left\{\frac{h_{2,10}}{h_2} + \frac{h_{2,1}}{h_2}\frac{h_{2,0}}{h_2}\right\}$$
$$+h_0^2h_2^2\left\{\frac{h_{2,11}}{h_2} + \left(\frac{h_{2,1}}{h_2}\right)^2 + 2\frac{h_{2,1}}{h_2}\frac{h_{0,1}}{h_0}\right\}$$

$$R_{23} = 0$$

$$R_{33} = R_{22}.$$

According to our two-component restriction, the spinor ϕ is of the form $\phi = \begin{pmatrix} u \\ -u \end{pmatrix}$, where $u = \begin{pmatrix} p \\ q \end{pmatrix}$. Then defining the γ matrices by (5.2), we obtain for $D_\mu = \bar{\phi}\gamma_\mu\phi$

$$D_0 = 2h_0(pp^* + qq^*), \qquad D_1 = \frac{2}{h_0}(p+q)(p+q)^*$$

$$D_2 = 2ih_2(pq^* - qp^*), \qquad D_3 = 2h_2(pp^* - qq^*).$$

The spinor connection operators Γ_ν for this metric are given by

$$\Gamma_0 = \frac{h_0h_{0,1}}{2}\beta_0\beta_1 - \frac{h_{0,3}}{2h_2}(\beta_3\beta_0 + \beta_3\beta_1) - \frac{h_{0,0}}{2h_0}\beta_1\beta_0 - \frac{h_{0,2}}{2h_2}(\beta_2\beta_0 + \beta_2\beta_1)$$

$$\Gamma_1 = -\frac{h_{0,1}}{2h_0}\beta_1\beta_0$$

$$\Gamma_2 = \frac{h_{2,0}}{2h_0}\beta_0\beta_2 + \frac{1}{2}\left(\frac{h_{2,0}}{h_0} - h_0h_{2,1}\right)\beta_1\beta_2 - \frac{h_{2,3}}{2h_2}\beta_3\beta_2 - \frac{h_{0,2}}{2h_0}\beta_1\beta_0$$

$$\Gamma_3 = \frac{h_{2,0}}{2h_0}\beta_0\beta_3 + \frac{1}{2}\left(\frac{h_{2,0}}{h_0} - h_0h_{2,1}\right)\beta_1\beta_3 - \frac{h_{2,2}}{2h_2}\beta_2\beta_3 - \frac{h_{0,3}}{2h_0}\beta_1\beta_0.$$

For the case $p = -q$ the neutrino condition (2.3) gives

$$(\ln h_0h_2^2p^2)_{,1} = 0$$

$$(\ln h_0h_2p^2)_{,2} - i(\ln h_0h_2p^2)_{,3} = 0;$$

the non-zero terms of the energy–momentum tensor are

$$E_{00} = 2ic\hbar h_0(pp^*_{,0} - p^*p_{,0})$$

$$E_{01} = ic\hbar h_0(pp^*_{,1} - p^*p_{,1})$$

$$E_{02} = ic\hbar h_0(pp^*_{,2} - p^*p_{,2}) - c\hbar h_0\frac{h_{2,3}}{h_2}pp^*$$

$$E_{03} = ic\hbar h_0(pp^*_{,3} - p^*p_{,3}) + c\hbar h_0\frac{h_{2,2}}{h_2}pp^*$$

and the vector H_μ is given by

$$H_\mu = -2\sqrt{2h_0 h_{2,1}}p^2(i\delta_\mu{}^2 + \delta_\mu{}^3).$$

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